1. (10 points) Two planes are orthogonal if their normal vectors are orthogonal. Find the equation of a plane that is orthogonal to the plane $3x - y + 2z = 1$ and that contains the points $(1, 1, 2)$ and $(2, 2, 1)$.

Find a normal vector $\vec{n}$ of the plane.

The plane contains $A = (1, 1, 2)$ and $B = (2, 2, 1)$

$\Rightarrow \vec{n}$ is perpendicular to $\overrightarrow{AB} = (1, 1, -1)$.

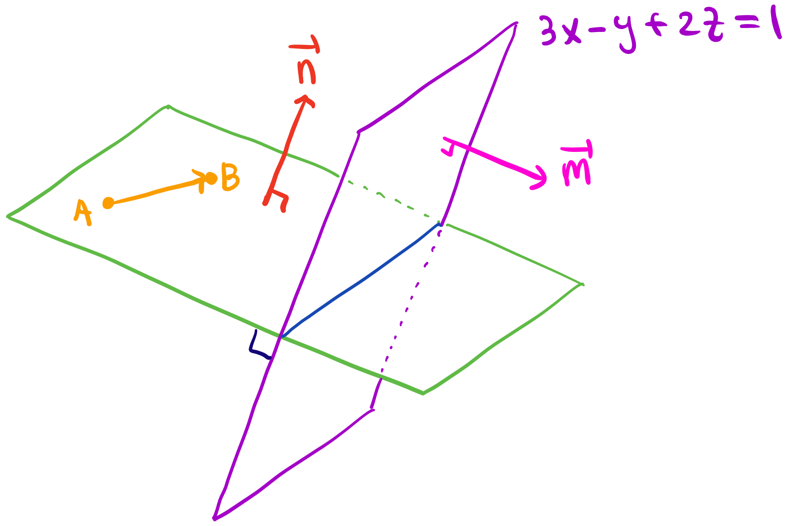
The plane is orthogonal to the other plane $3x - y + 2z = 1$.

$\Rightarrow \vec{n}$ is perpendicular to a normal vector $\vec{m} = (3, -1, 2)$ of the other plane.

$\rightsquigarrow \vec{n} = \overrightarrow{AB} \times \vec{m} = (1, 1, -1) \times (3, -1, 2) = (1, -5, -4)$

The plane equation is

$$\boxed{1 \cdot (x-1) - 5 \cdot (y-1) - 4(z-2) = 0}$$



Note The answer can be given in many other forms, such as $1 \cdot (x-2) - 5 \cdot (y-2) - 4 \cdot (z-1) = 0$ or

$x - 5y - 4z + 12 = 0$.

2. Parts (a) and (b) are about a plane and part (c) is about a line.

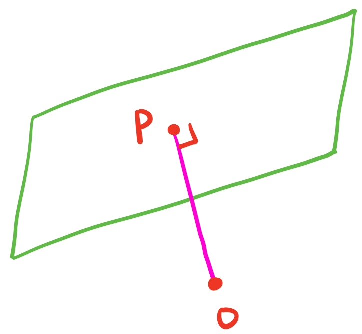
(a) (2 points) Find the distance of the origin from the plane $2x - y - 2z = 6$.

The distance from $(0,0,0)$ to the plane is

$$\frac{|2\cdot 0 - 0 - 2\cdot 0 - 6|}{\sqrt{2^2+(-1)^2+(-2)^2}} = \frac{6}{3} = \boxed{2}$$

(b) (4 points) Find the point on the plane $2x - y - 2z = 6$ that is closest to the origin

Denote this point by P.



$\overrightarrow{OP}$ is perpendicular to the plane.

A normal vector of the plane is

$\vec{n} = (2, -1, -2)$

$\Rightarrow \overrightarrow{OP}$ is parallel to $\vec{n}$.

$\Rightarrow \overrightarrow{OP} = \lambda \vec{n} = \lambda(2,-1,-2) = (2\lambda, -\lambda, -2\lambda)$ for some λ.

P is on the plane $2x - y - 2z = 6$

$\Rightarrow 2\cdot 2\lambda - (-\lambda) - 2(-2\lambda) = 6$

$\rightsquigarrow 9\lambda = 6 \rightsquigarrow \lambda = \frac{2}{3}$

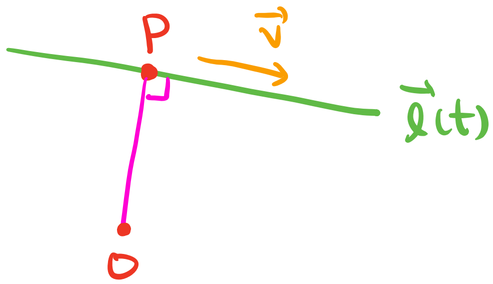
$\Rightarrow P = \boxed{\left(\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)}$

(c) (4 points) Find the point on the line $x = 2t - 1$, $y = t + 2$, $z = -t - 6$ that is closest to the origin.

Denote this point by P.

The line equation is $\vec{\ell}(t) = (2t-1, t+2, -t+6)$.

A direction vector is $\vec{v} = (2, 1, -1)$.



$\overrightarrow{OP}$ and $\vec{v}$ must be perpendicular

$\Rightarrow$ We solve $\vec{\ell}(t) \cdot \vec{v} = 0$

$\rightsquigarrow (2t-1, t+2, -t-6) \cdot (2, 1, -1) = 0$

$\rightsquigarrow 2(2t-1) + 1 \cdot (t+2) - (-t-6) = 0$

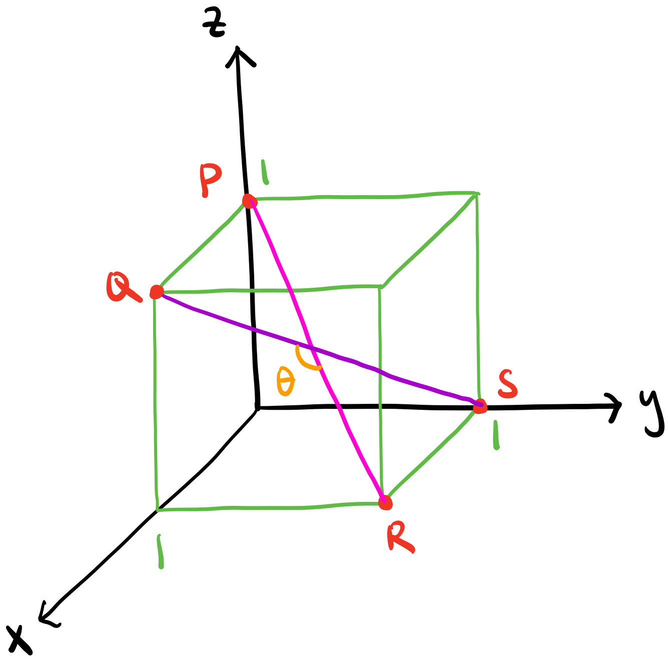
$\rightsquigarrow 6t + 6 = 0 \rightsquigarrow t = -1$

$\Rightarrow P = \vec{\ell}(-1) = \boxed{(-3, 1, 5)}$

Note Alternatively, you can find the minimum of

$$|\vec{\ell}(t)|^2 = (2t-1)^2 + (t+2)^2 + (-t-6)^2$$

3. (10 points) A diagonal of a cube joins a vertex to the opposite vertex. Therefore, if each edge is of length 1, the length of the diagonal is $\sqrt{3}$. If θ is the angle between two distinct diagonals, find $\cos\theta$.



$$\begin{cases} P = (0,0,1) \\ Q = (1,0,1) \\ R = (1,1,0) \\ S = (0,1,0) \end{cases}$$

The diagonal vectors are

$\overrightarrow{PR} = (1,1,-1)$, $\overrightarrow{QS} = (-1,1,-1)$

$$\cos\theta = \frac{\overrightarrow{PR}\cdot\overrightarrow{QS}}{|\overrightarrow{PR}||\overrightarrow{QS}|}$$

$$\overrightarrow{PR}\cdot\overrightarrow{QS} = 1\cdot(-1) + 1\cdot 1 + (-1)\cdot(-1) = 1.$$

$$|\overrightarrow{PR}| = \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}$$

$$|\overrightarrow{QS}| = \sqrt{(-1)^2+1^2+(-1)^2} = \sqrt{3}.$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{3}\cdot\sqrt{3}} = \boxed{\frac{1}{3}}$$

Note The statement of the problem is slightly ambiguous, because you can also get $\cos\theta = -\frac{1}{3}$ by considering $\overrightarrow{SQ}$ instead of $\overrightarrow{QS}$.

4. Consider the paraboloid $z = x^2 + y^2$.

(a) (5 points) Find the equation of the plane that is tangent to the paraboloid at the point $(1, 1, 2)$.

The paraboloid is the graph of $f(x,y) = x^2 + y^2$.

The tangent plane equation is

$$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$f_x = 2x \rightsquigarrow f_x(1,1) = 2.$

$f_y = 2y \rightsquigarrow f_y(1,1) = 2.$

$$\Rightarrow \boxed{z = 2 + 2(x-1) + 2(y-1)}$$

(b) (5 points) The normal line to the tangent plane at the point $(1, 1, 2)$ intersects the paraboloid at another point. Find that point.

The tangent plane equation can be written as

$$2(x-1) + 2(y-1) - (z-2) = 0$$

A normal vector is $\vec{n} = (2, 2, -1)$.

$\Rightarrow$ The normal line is $\vec{\ell}(t) = (1+2t, 1+2t, 2-t)$.

At the intersection:

$$z = x^2 + y^2 \rightsquigarrow 2-t = (1+2t)^2 + (1+2t)^2$$

$$\rightsquigarrow 2-t = 2 + 8t + 8t^2$$

$$\rightsquigarrow 8t^2 + 9t = 0 \rightsquigarrow t = 0, -\frac{9}{8}.$$

$$\Rightarrow \vec{\ell}(0) = (1, 1, 2),\ \vec{\ell}\left(-\frac{9}{8}\right) = \boxed{\left(-\frac{5}{4}, -\frac{5}{4}, \frac{25}{8}\right)}$$

5. Each part asks you to find a partial derivative.

(a) (3 points) If $f(x, y) = x^2 + y^2 - xy$ and $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial f}{\partial r}$.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

Chain rule

$$\frac{\partial f}{\partial x} = 2x - y, \quad \frac{\partial f}{\partial y} = 2y - x.$$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\Rightarrow \frac{\partial f}{\partial r} = (2x - y)\cos\theta + (2y - x)\sin\theta$$

$$= \boxed{(2r\cos\theta - r\sin\theta)\cos\theta + (2r\sin\theta - r\cos\theta)\sin\theta}$$

$x = r\cos\theta$, $y = r\sin\theta$

(b) (3 points) If $x^2 + y^2 + z^2 = xy\sin(z)$ defines z as a function of x and y, find $\frac{\partial z}{\partial x}$.

$$x^2 + y^2 + z^2 - xy\sin(z) = 0$$

Set $f(x, y, z) = x^2 + y^2 + z^2 - xy\sin(z)$

$$f_x = 2x - y\sin(z), \quad f_z = 2z - xy\cos(z)$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = \boxed{\frac{-2x + y\sin(z)}{2z - xy\cos(z)}}$$

Implicit function theorem

(c) (4 points) If $z = f(x, y)$ and $x = r + 2s$, $y = 2r - s$, find $\frac{\partial^2 z}{\partial r^2}$.

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

Chain rule

$$\frac{\partial x}{\partial r} = 1, \quad \frac{\partial y}{\partial r} = 2 \qquad (*)$$

$$\Rightarrow \frac{\partial z}{\partial r} = f_x + 2f_y$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}(f_x + 2f_y) = \frac{\partial f_x}{\partial r} + \frac{\partial f_y}{\partial r}$$

$$\frac{\partial f_x}{\partial r} = \frac{\partial f_x}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y} \cdot \frac{\partial y}{\partial r} = f_{xx} + 2f_{xy}$$

Chain rule $\qquad (*)$

$$\frac{\partial f_y}{\partial r} = \frac{\partial f_y}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y} \cdot \frac{\partial y}{\partial r} = f_{yx} + 2f_{yy}.$$

$$\Rightarrow \frac{\partial^2 z}{\partial r^2} = (f_{xx} + 2f_{xy}) + 2(f_{yx} + 2f_{yy})$$

$$= \boxed{f_{xx} + 4f_{xy} + 4f_{yy}}$$

$f_{xy} = f_{yx}$

6. Consider the space curve $\mathbf{r}(t) = (\cos t, \sin t, 2t^{3/2}/3)$.

(a) (4 points) Find the length of the space curve from $t = 0$ to $t = \pi$.

Arc length $= \int_0^{\pi} |\vec{r}'(t)|\, dt$

$\vec{r}'(t) = (-\sin t, \cos t, t^{1/2})$

$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + t} = \sqrt{1+t}$.

$\Rightarrow$ Arc length $= \int_0^{\pi} \sqrt{1+t}\, dt = \frac{2}{3}(1+t)^{3/2} \Big|_{t=0}^{t=\pi}$

$= \boxed{\frac{2}{3}\left((1+\pi)^{3/2} - 1\right)}$

(b) (3 points) Find the equation of the tangent line to the space curve at $t = 0$.

The tangent vector is $\vec{r}'(0) = (0, 1, 0)$

The point of tangency is $\vec{r}(0) = (1, 0, 0)$

$\Rightarrow$ The tangent line is parametrized by

$\vec{\ell}(t) = (1 + 0\cdot t, 0 + 1\cdot t, 0 + 0\cdot t) = \boxed{(1, t, 0)}$

(c) (3 points) If t is time, what is the speed of the particle at $t = 2\pi$?

The speed at $t = 2\pi$ is $|\vec{r}'(2\pi)| = \boxed{\sqrt{1+2\pi}}$

7. Consider the function $f(x, y) = \frac{x^2}{4} + \frac{y^2}{9}$.

(a) (3 points) Find the unit vector in the direction of fastest increase at $x = 2$, $y = 3$.

The direction is given by $\nabla f(2,3)$.

$$\nabla f = (f_x, f_y) = \left(\frac{x}{2}, \frac{2y}{9}\right)$$

$$\nabla f(2,3) = \left(\frac{2}{2}, \frac{2\cdot 3}{9}\right) = \left(1, \frac{2}{3}\right).$$

$$\Rightarrow \frac{\nabla f(2,3)}{|\nabla f(2,3)|} = \frac{(1, 2/3)}{\sqrt{1^2 + (2/3)^2}} = \boxed{\frac{1}{\sqrt{13}}(3,2)}$$

(b) (4 points) Find the unit vector along which the directional derivative is zero at $x = 2$, $y = 3$.

Set $\vec{u} = (a, b)$

$$D_{\vec{u}} f(2,3) = \nabla f(2,3) \cdot \vec{u} = \left(1, \frac{2}{3}\right) \cdot (a, b)$$

$$= a + \frac{2}{3} b = 0$$

Take $a = -2$, $b = 3$

$$\Rightarrow \vec{u} = \frac{(-2,3)}{|(-2,3)|} = \frac{(-2,3)}{\sqrt{(-2)^2 + 3^2}} = \boxed{\frac{1}{\sqrt{13}}(-2,3)}$$

(c) (3 points) Find the directional derivative along the unit vector $\mathbf{u} = 3\mathbf{i}/5 + 4\mathbf{j}/5$.

$$D_{\vec{u}} f(3,2) = \nabla f(3,2) \cdot \vec{u} = \left(1, \frac{2}{3}\right) \cdot \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$= 1 \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{4}{5} = \boxed{\frac{17}{15}}$$

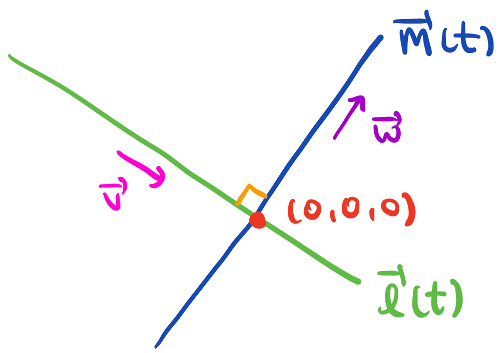
8. Consider the line $(x, y, z) = (t, 2t, -2t)$, with t being the parameter.

(a) (3 points) Find the parametric equation of another line which passes through the origin and intersects the given line at 90^o.

The given line $\vec{\ell}(t) = (t, 2t, -2t)$ has a direction vector $\vec{v} = (1, 2, -2)$.

Also, it passes through the origin at $t=0$.

Set $\vec{w} = (a, b, c)$ to be a direction vector of the desired line $\vec{m}(t)$



$\vec{v}$ and $\vec{w}$ must be perpendicular

$\rightsquigarrow \vec{v} \cdot \vec{w} = 0 \rightsquigarrow (1, 2, -2) \cdot (a, b, c) = 0$

$\rightsquigarrow a + 2b - 2c = 0$

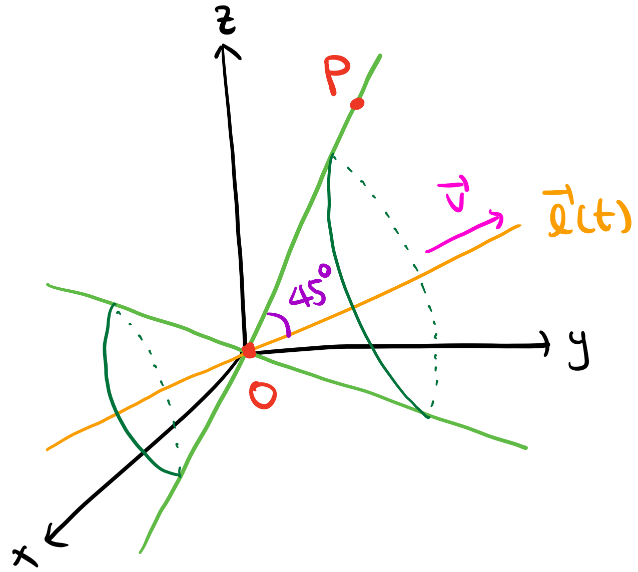
Take $a = 0,\ b = 1,\ c = 1 \Rightarrow \vec{w} = (0, 1, 1)$

$\Rightarrow \vec{m}(t) = (0 + 0 \cdot t,\ 0 + 1 \cdot t,\ 0 + 1 \cdot t) = \boxed{(0, t, t)}$

<u>Note</u> There are infinitely many possible answers, depending on the choice of a, b, c.

(b) (7 points) Now consider the cone whose axis is the given line and whose vertex is at the origin. In addition, every line on the cone joining the vertex to some other point on the cone makes an angle of 45^o with the axis (or the given line). Find the equation of that cone.

Very tricky!



For every point $P=(x,y,z)$ on the cone, $\vec{OP}$ and $\vec{v}$ form an angle of $\frac{\pi}{4}$ $(=45^\circ)$.

$\Rightarrow \vec{OP}\cdot\vec{v} = |\vec{OP}||\vec{v}|\cos(\frac{\pi}{4})$

$\vec{OP} = (x,y,z)$, $\vec{v} = (1,2,-2)$

$\vec{OP}\cdot\vec{v} = x+2y-2z$

$|\vec{OP}| = \sqrt{x^2+y^2+z^2}$, $|\vec{v}| = \sqrt{1^2+2^2+(-2)^2} = 3$.

$\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

$\Rightarrow x+2y-2z = \frac{3}{\sqrt{2}}\sqrt{x^2+y^2+z^2}$.

$\Rightarrow \boxed{2(x+2y-2z)^2 = 9(x^2+y^2+z^2)}$